Echoes in classical dynamical systems

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Abstract
Echoes arise when external manipulations to a system induce a reversal of
its time evolution that leads to a more or less perfect recovery of the initial
state. We discuss the accuracy with which a cloud of trajectories returns to the
initial state in classical dynamical systems that are exposed to additive noise,
and small differences in the equations of motion for forward and backward
evolution. The cases of integrable and chaotic motion, and small or large noise
are studied in some detail and many different dynamical laws are identified.
Experimental tests in two-dimensional flows that show chaotic advection are
proposed.

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1. Introduction

Echoes arise when through suitable manipulations in a system the dynamics is reversed and
a more or less complete recovery of the initial state is achieved. Acoustical echoes arise
from reflections of sound at walls, spin echoes from reversals of magnetic fields (Hahn 1950,
Carr and Purcell 1954), current echoes through a sequence of suitable electromagnetic pulses
(Niggemeier *et al* 1993) and Loschmidt echoes from a reversal of momenta in a Hamiltonian
system (Loschmidt 1876). That echoes can also appear in many particle systems is at first
surprising since it seems to be in conflict with the irreversibility implied by the second law of
thermodynamics. Closer inspection shows, however, that the recovery of the initial state is not
perfect and studies of the deviations tell a lot about the mechanisms that break reversibility.

Several aspects of echo phenomena in dynamical systems have recently been studied
in connection with Loschmidt echoes in quantum systems (triggered by Pastawski *et al*
(1995) and Levstein *et al* (1998)). An initial state $|0\rangle$ is propagated forward in time
with Hamiltonian H and then back in time with a slightly different Hamiltonian H'
(as suggested by Peres (1984)). The loss of coherence is measured in terms of the
fidelity $\langle 0|e^{iH't/\hbar}\,e^{-iHt/\hbar}|0\rangle$. This is the same as calculating the overlap between the states
$|t\rangle = e^{-iHt/\hbar}|0\rangle$ and $|t'\rangle = e^{-iHt'/\hbar}|0\rangle$, obtained by propagating the same initial state $|0\rangle$ for

the same time t but with two different Hamiltonians. The decay of the overlap as a function of time and difference in Hamiltonian and the various time regimes have been the subject of several recent papers (Jalabert and Pastawski 2001, Jacquod *et al* 2001, Cerrutti and Tomsovic 2002, Benenti and Casati 2002, Wisniacki and Cohen 2001, Prosen and Znidaric 2002, Prosen and Seligman 2002).

The present paper is devoted to Loschmidt echoes in classical dynamical systems. If there is no difference between forward and backward equations of motion, the initial state is recovered perfectly. But what happens if there are small differences or if the system is exposed to noise? And what are the differences between echo experiments in integrable and chaotic systems? These questions will be addressed for Gaussian densities in linearized flows: they provide a convenient and sufficiently general class of densities in which a large variety of dynamical behaviour can be identified.

Besides the obvious connection to the quantum echo experiments, the calculations are of some relevance for two other directly classical situations: numerical trajectory reversals and reversibilty in advection.

When the equations of motion are reversed in numerical calculations, trajectories will typically not return to their starting point. For chaotic systems, the inherent sensitivity to initial conditions suggests an exponentially large deviation. A lack of growth has been used as an indicator for quantum regularity (Casati *et al* 1986). Obviously, no such problems should arise for perfect reversals and perfect numerical integrators since the solutions to the equations of motion are uniquely specified by the initial conditions (barring singular points in the differential equations). The fact that trajectories do not return to their starting points thus reflects numerical inaccuracies from finite time steps and limited resolution. Using Gaussians to represent a cloud of initial conditions, noise to reflect truncation errors and differences between forward and backward integration algorithms, this numerical reversibility experiment can be connected to the problem considered here.

A popular and impressive demonstration of echoes in classical systems is provided by flow reversals in viscous liquids: a blob of dye can be stretched out until it is barely visible, but upon reversal of the flow it re-forms almost completely! The multimedia fluid mechanics CD (Homsey *et al* 2000) contains several demonstrations in laminar flows. The connection to chaos comes through experiments on chaotic advection (Aref 1984, 2002, Ottino 1990). For instance, in their experiments on chaotic advection, Chaiken *et al* (1986) noted that if the dye passed through a chaotic region the recovery was less perfect, but they did not investigate this in detail. Since many of the typical flows can be realized experimentally, the dynamics of the Gaussians discussed below should also be experimentally accessible.

The types of systems considered here are classical. They may be exposed to additive white noise and forward and backward motion may differ. The state of the system is characterized by a smooth density in phase space, and the evolution equation is the Fokker–Planck equation with appropriate drift term. The discussion will be limited to densities that are Gaussian in shape, for which a reasonably complete general discussion is possible. Superpositions of such Gaussians can be used to approximate other densities. General expressions in arbitrary dimensions will be given, but their implications are most evident in two-dimensional (2D) conservative flows: this is also the case that is accessible in hydrodynamical systems.

The outline of the paper is as follows. In section 2, the dynamics of the centre of mass and the variances for Gaussian densities will be discussed. In section 3, this information is applied to the discussion of echoes in integrable systems (section 3.1) and chaotic ones (section 3.2). Section 4 contains a discussion of the results and some remarks on experimental tests in 2D advection systems.

2. Gaussian densities

2.1. Outline of echo experiments

The calculation of an echo naturally divides into two steps: the forward evolution up to some time T under one set of equations, followed by the backward evolution under a perhaps slightly different set of equations for the same time interval. If a mapping of initial conditions under a class of time evolutions can be found, say $\rho_f = U_1(T)\rho_i$ for the forward evolution under flow $\mathbf{u}_1$, then we can write for the backward evolution with flow $\mathbf{u}_2$ the formal expression $\rho_b = U_2(-T)\rho_f$, so that the mapping to the echo state ρ_e becomes

$$\rho_e = U_2(-T)U_1(T)\rho_i. \tag{1}$$

Thus, on the technical level it suffices to find U for the forward dynamics of a sufficiently large class of densities and systems. Note that in the presence of noise or in a dissipative system there is a difference between (i) comparing the initial state with the state obtained by propagating an initial condition over the complete cycle of forward and backward evolution and (ii) comparing the states obtained by propagating the same initial condition forward in time with two different flows: dissipation and noise break the reversibility that permitted the change in protocol in the quantum case.

The evolution equation for the densities $\rho(\mathbf{x}, t)$ is the Fokker–Planck equation,

$$\dot{\rho} = -\nabla(\mathbf{u}\rho) + D\Delta\rho \tag{2}$$

where D is the molecular diffusion constant. Echoes are most easily identified when initial and final densities are sufficiently similar, as is the case for strongly localized objects. More complicated initial densities may be approximated by superpositions of localized ones. The dynamics for localized densities splits, in leading order in moments, into two parts, the motion of the centre of mass and the changes in shape and size, as measured by the variances. This expansion may be extended to higher order moments of the density, but the equations become too cumbersome to analyse.

2.2. Centre of mass motion

The centre of mass of a localized density follows a classical trajectory $\mathbf{x}_P(t)$, where

$$\dot{\mathbf{x}}_P(t) = \mathbf{u}(\mathbf{x}_P(t), t). \tag{3}$$

The notation here is borrowed from hydrodynamic advection (Aref 1984, 2002), where $\mathbf{u}$ is a velocity field and $\mathbf{x}_P$ the trajectory of a particle advected by the fluid. In other situations, the velocity field $\mathbf{u}(\mathbf{x}, t)$ has to be replaced by the right-hand side $\mathbf{f}$ of an evolution equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

For the backward propagation, where a small modification of the flow field is permitted, the trajectory may differ from the one during forward evolution and we need to estimate the differences between the two. Let $\mathbf{x}_P(t)$ be a trajectory in a velocity field $\mathbf{u}$, and $\mathbf{x}_P(t) + \mathbf{q}(t)$ one in the perturbed velocity field $\mathbf{u} + \delta\mathbf{u}$. To first order in $\mathbf{q}$, the equation becomes

$$\dot{\mathbf{q}}(t) = A(t)\mathbf{q}(t) + \delta\mathbf{u}(\mathbf{x}_P(t), t) \tag{4}$$

where A is the linearization of the full velocity field $\mathbf{u} + \delta\mathbf{u}$ at the trajectory $\mathbf{x}_P$,

$$A_{ij} = \frac{\partial(u_i + \delta u_i)}{\partial x_j}(\mathbf{x}_P(t), t). \tag{5}$$

With the help of the monodromy matrix $M(t)$, the solution to

$$\dot{M} = AM \tag{6}$$

with initial condition $M(0) = 1$, a formal solution can be given,

$$\mathbf{q}(t) = M(t)\left(q(0) + \int_0^t d\tau\, M^{-1}(\tau)\delta\mathbf{u}(\mathbf{x}_P(\tau), \tau)\right). \tag{7}$$

This is the general solution for the displacement in a perturbed velocity field. A discussion of specific examples will be deferred to section 3.

2.3. Variances

With $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_P(t)$ the coordinates relative to the centre of mass, the density can be written as $\rho(\tilde{\mathbf{x}}, t) = \rho(\mathbf{x} - \mathbf{x}_P(t), t)$ and the Fokker–Planck equation becomes

$$\dot{\rho}(\tilde{\mathbf{x}}, t) = -\nabla(\mathbf{u}(\mathbf{x}_P + \tilde{\mathbf{x}}, t)\rho) + (\mathbf{u}(\mathbf{x}_P, t) \cdot \nabla)\rho + D\Delta\rho, \tag{8}$$

where now all spatial derivatives are with respect to the relative coordinate $\tilde{\mathbf{x}}$. The localization of the densities allows a linearization of the velocity field near the trajectory, i.e.,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}_P(t), t) + A(t)\tilde{\mathbf{x}} \tag{9}$$

with the derivative matrix A, equation (5). Then

$$\dot{\rho} = -(\text{tr}\, A)\rho - ((\tilde{\mathbf{x}}^T A^T) \cdot \nabla)\rho + D\Delta\rho. \tag{10}$$

For conservative Hamiltonian systems and incompressible flows, the trace of A vanishes; the discussion will henceforth be limited to that case. The next step in the analysis is to note that the equations are second order with a linear position dependence at most, so that a solution in terms of Gaussian densities is possible (Eckhardt 1990). In an n-dimensional phase space with variance matrix $\Gamma(t)$, they are given by

$$\rho(\tilde{\mathbf{x}}, t) = \pi^{-n/2}(\det\Gamma)^{-1/2}e^{-\tilde{\mathbf{x}}^T\Gamma^{-1}\tilde{\mathbf{x}}} \tag{11}$$

where Γ satisfies

$$\dot{\Gamma} = 2D + A\Gamma + \Gamma A^T \tag{12}$$

(note that Γ as a kernel for a quadratic form is symmetric). With the help of the monodromy matrix $M(t)$ a closed expression for the variance matrix can be given,

$$\Gamma(t) = M(t)\left(\Gamma_i + 2D\int_0^t (M(\tau)^T M(\tau))^{-1}\, d\tau\right)M(t)^T. \tag{13}$$

This is the central formula for the dynamics of the variances on which the calculation of the various situations can be based. For the specific cases studied in section 3, a direct solution of (12) was found to be simpler.

3. Echoes

3.1. Classical fidelity

The analysis of echoes now proceeds as follows: we start with an initial density ρ_i with variance matrix Γ_i. This density evolves under the influence of a velocity field $\mathbf{u}$ and additive white noise for some time T. At the end of this time interval, the position of the density is $\mathbf{x}_P(T)$ and the variance is $\Gamma_f(T)$, as given by (13). Then the field is reversed. If the reversal is perfect, the new velocity field $\mathbf{u}'$ equals $-\mathbf{u}$ and the centre of mass returns exactly. If there is a slight deviation, the centre of mass will move along a trajectory with a small displacement $\mathbf{q}(t)$ according to (7). In both cases, however, the variance does not return exactly, unless all

noise is suppressed, i.e., $D = 0$. This holds quite generally, the only approximation being the Gaussian shape of the density and linearization of the velocity field near the trajectory.

A general Gaussian echo may then be displaced with its centre of mass by $\mathbf{q}$ and will have a variance matrix Γ_e. For a comparison between initial and echo densities of states, the positions and variances can be used directly. But it is also possible to mimic the quantum fidelity expression and introduce an overlap between classical phase-space densities. The main difference is that quantum wavefunctions are normalized within the L^2 norm and classical densities are not. The proper definition of the fidelity as the cosine of the angle between the two densities in Hilbert space is then

$$\mathcal{O} = \frac{\int d\mathbf{x}\, \rho_i(\mathbf{x})\rho_e(\mathbf{x})}{\left(\int d\mathbf{x}\, \rho_i(\mathbf{x})^2\right)^{1/2} \left(\int d\mathbf{x}\, \rho_e(\mathbf{x})^2\right)^{1/2}}. \tag{14}$$

Without the normalization, the overlap could change even if both densities evolve in the same way. Thus the definition given by Prosen and Znidaric (2002) has to be modified. For the case of two Gaussians, an initial density ρ_i centred at zero and with variances Γ_i and an echo density

$$\rho_e = \pi^{-n/2} (\det \Gamma_e)^{-1/2} e^{-(\mathbf{x}-\mathbf{q})^T \Gamma_e^{-1} (\mathbf{x}-\mathbf{q})} \tag{15}$$

the overlap becomes

$$\mathcal{O} = 2^{n/2} \sqrt{\frac{\sqrt{\det \Gamma_i} \sqrt{\det \Gamma_e}}{\det(\Gamma_i + \Gamma_e)}}\, e^{-\mathbf{q}^T (\Gamma_i^{-1} + \Gamma_e^{-1})\mathbf{q}}. \tag{16}$$

The overlap integral indicates two very different kinds of contributions: the prefactor measures the reduction in overlap by the spreading of the density. The exponential factor accounts for the rapid drop-off in overlap when the centres of the Gaussians are separated; the Gaussian form is clearly connected to the Gaussian tails in the density and would be different, e.g., for exponential tails in the density.

3.2. Shear flows

In laminar flows, neighbouring trajectories see slightly different velocities and separate linearly in time. When combined with noise, a cubic growth of the variance results. Specifically, consider a 2D shear flow

$$\mathbf{u} = \begin{pmatrix} \alpha y \\ 0 \end{pmatrix} \tag{17}$$

of shear rate α. The associated monodromy matrix is

$$M(t) = \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix}. \tag{18}$$

During forward evolution the variances become

$$\Gamma_{11}^{(f)} = \Gamma_{11}^{(i)} + 2\alpha \Gamma_{12}^{(i)} T + \alpha^2 \Gamma_{22}^{(i)} T^2 + 2DT + \tfrac{2}{3}\alpha^2 DT^3 \tag{19}$$

$$\Gamma_{12}^{(f)} = \Gamma_{12}^{(i)} + \alpha \Gamma_{22}^{(i)} T + \alpha DT^2 \tag{20}$$

$$\Gamma_{22}^{(f)} = \Gamma_{22}^{(i)} + 2DT. \tag{21}$$

The T^3 contribution to the variances has also been discussed by Rhines and Young (1983) in the context of fluid mixing.

For the reversal, we allow for a different shear rate α' and some perturbation in the velocity field. If $\mathbf{q}_f$ is an initial displacement in the trajectory and $\delta\mathbf{u} = (u_1, u_2)$ a constant perturbation to the velocity field, the displacement of the echo will be

$$q_{1,e} = q_{1,f} - (u_1 - \alpha'q_{2,f})t - \alpha'u_2t^2/2 \tag{22}$$

$$q_{2,e} = q_{2,f} - \alpha'u_2t - \alpha'u_2t^2/2. \tag{23}$$

The variances become

$$\Gamma_{11}^{(e)} = \Gamma_{11}^{(i)} + 2(\alpha - \alpha')\Gamma_{12}^{(i)}T + (\alpha - \alpha')^2\Gamma_{22}^{(i)}T^2 + 4DT + \tfrac{1}{3}(2\alpha^2 + 8\alpha'^2 - 6\alpha\alpha')DT^3 \tag{24}$$

$$\Gamma_{12}^{(e)} = \Gamma_{12}^{(i)} + (\alpha - \alpha')\Gamma_{22}^{(i)}T + (\alpha - 3\alpha')DT^2 \tag{25}$$

$$\Gamma_{22}^{(e)} = \Gamma_{22}^{(i)} + 4DT. \tag{26}$$

This result for the variances may be verified for a few limiting cases. (i) Without shear $\alpha = \alpha' = 0$, the diagonal elements increase like $4DT$ as for regular diffusion over a time interval $2T$. (ii) Without diffusion ($D = 0$), the determinant of the matrix does not change. (iii) Without diffusion ($D = 0$) and equal shear in the forward and backward directions ($\alpha = \alpha'$), the reversal is perfect and the initial variances are restored. (iv) The parameters $\alpha' = -\alpha$ correspond to the situation that the backward integration is just a continuation of the forward integration and the expressions (24)–(26) agree with (19)–(21) for an evolution time of $2T$. (v) For equal shear $\alpha = \alpha'$ in forward and backward directions, the variances are

$$\Gamma_{11}^{(e)} = \Gamma_{11}^{(i)} + 4DT + \tfrac{4}{3}\alpha^2DT^3 \tag{27}$$

$$\Gamma_{12}^{(e)} = \Gamma_{12}^{(i)} - 2\alpha DT^2 \tag{28}$$

$$\Gamma_{22}^{(e)} = \Gamma_{22}^{(i)} + 4DT; \tag{29}$$

the prefactor of the cubic term in $\Gamma_{11}^{(e)}$ is smaller than that obtained from (24) for a time $2T$, indicating that the reversal of the shear induced broadening is partial and not complete.

The different time regimes in the variances are easily identified. First, consider the terms linear in time: diffusion will be noticable on a time scale $T_D \approx 1/D$, the differences in the shear rates on a time scale $T_\delta \approx 1/|\alpha - \alpha'|$. Thus, a large diffusion can swamp the effects from the difference between the two Hamiltonians. For the nonlinear terms, the one with the difference in shear rates appears around T_δ, and the one with the cubic term in diffusion near $1/\alpha$. In typical applications the latter should be smaller than T_δ.

The classical fidelity contains the determinants of the variances. In the absence of diffusion, $D = 0$, we have $\det\Gamma_e = \det\Gamma_i$ and

$$\det(\Gamma_i + \Gamma_e) = 4\det\Gamma_i + (\alpha - \alpha')^2T^2\Gamma_{22}^{(i)} \tag{30}$$

so that the fidelity has a $1 - \text{const} \cdot T^2$ behaviour for short times and a $1/(|\alpha - \alpha'|T)$ decay for times longer than T_δ. With diffusion and equal forward and backward shears, $\alpha = \alpha'$, we have

$$\det\Gamma_e = \det\Gamma_i + 4\big(\Gamma_{11}^{(i)} + \Gamma_{22}^{(i)}\big)DT + 16D^2T^2 + 4\Gamma_{12}^{(i)}\alpha DT^2 + \tfrac{4}{3}\Gamma_{12}^{(i)}\alpha^2DT^3 + \tfrac{4}{3}\alpha^2D^2T^4 \tag{31}$$

and

$$\det(\Gamma_i + \Gamma_e) = 4\det\Gamma_i + 8\big(\Gamma_{11}^{(i)} + \Gamma_{22}^{(i)}\big)DT$$
$$+ 16D^2T^2 + 8\Gamma_{12}^{(i)}\alpha DT^2 + \tfrac{8}{3}\Gamma_{12}^{(i)}\alpha^2DT^3 + \tfrac{4}{3}\alpha^2D^2T^4 \tag{32}$$

so that a short-time behaviour $1 - \text{const} \cdot T^2$ and a long-time behaviour of $1/(\sqrt{\alpha D}T)$ follow.

The contributions from displacements enter the exponent and can, in principle, introduce rapid decays. Consider, e.g., the case $\alpha = \alpha'$ and weak diffusion, weak shear and short times, so that DT and αT are smaller than the initial variances. Then $\Gamma^{(e)} \approx \Gamma^{(i)}$ and the only time dependence will come from the perturbations $\mathbf{q}_0$ and δu: the exponent will contain a polynominal of fourth order in time, and this leads to a rapid and faster than exponential decay. If diffusion is added, the increase in variance can compensate part of the displacement growth, but for strong diffusion and large times, an exponential decay will remain: the square of $\mathbf{q}^2$ will increase like T^4 and the variance increases like T^3, so that the ratio increases linearly, giving an exponential decay.

Thus, without displacement the overlap integral decays algebraically as contained in the prefactor, but with displacement the decrease can be dramatic when the densities are separated by more than their widths.

3.3. Chaotic systems

As a model for a chaotic system, take a simple hyperbolic motion, $\mathbf{u} = (\lambda x, -\lambda y)$, and assume an initial density aligned with the unstable (x-) and stable (y-) directions, i.e., $\Gamma_{12}^{(i)} = 0$. After forward evolution we have

$$\Gamma_{11}^{(f)} = \Gamma_{11}^{(i)} e^{2\lambda T} + \frac{D}{\lambda}(e^{2\lambda T} - 1) \tag{33}$$

$$\Gamma_{22}^{(f)} = \Gamma_{22}^{(i)} e^{-2\lambda T} + \frac{D}{\lambda}(1 - e^{-2\lambda T}). \tag{34}$$

Thus, there is an exponential contraction down to the limit set by diffusion. If the backward integration has a slightly different stretching rate λ', then for the echo

$$\Gamma_{11}^{(e)} = \Gamma_{11}^{(i)} e^{2(\lambda-\lambda')T} + \frac{D}{\lambda}(e^{2\lambda T} - 1) e^{-2\lambda'T} + \frac{D}{\lambda'}(1 - e^{-2\lambda'T}) \tag{35}$$

$$\Gamma_{22}^{(e)} = \Gamma_{22}^{(i)} e^{-2(\lambda-\lambda')T} + \frac{D}{\lambda}(1 - e^{-2\lambda T}) e^{2\lambda'T} + \frac{D}{\lambda'}(e^{2\lambda'T} - 1). \tag{36}$$

As in the previous case we can study various limiting situations, such as $\lambda, \lambda' \to 0$, where linear diffusion results, or $\lambda = -\lambda'$, where the expressions (33) and (34) for times up to $2T$ are recovered. When forward and backward stretching rates are the same, $\lambda = \lambda'$, the variances become

$$\Gamma_{11}^{(e)} = \Gamma_{11}^{(i)} + 2\frac{D}{\lambda}(1 - e^{-2\lambda T}) \tag{37}$$

$$\Gamma_{22}^{(e)} = \Gamma_{22}^{(i)} + 2\frac{D}{\lambda'}(e^{2\lambda'T} - 1). \tag{38}$$

Note that the variance in x has hardly changed whereas the one in y grows exponentially. The reason is that the x-variance grows during the forward integration and then collapses during the backward evolution, down to a limit set by the diffusional broadening. The exponential growth and contraction is, thus, almost perfectly compensated. For the y-variance we have, first, the contraction, down to the limit set by diffusion. The expansion during the backward evolution then starts from this finite amplitude, and not from the exponentially small contraction of the deterministic evolution of the initial variance. As a result, the echo is stretched out along the direction that was the stable one during forward evolution. If forward and backward evolution are interchanged, then so is the orientation of the spreading of the density: it will then point in the x-direction. For this growth to be noticeable, the time evolution has to be followed for times longer than about $(\ln(\lambda\Gamma^{(i)}/D))/(2\lambda)$.

The behaviour of the classical fidelity for short times is quadratic or linear, for $D = 0$ and $D \neq 0$, respectively. For longer times, there is an exponential decay like $\exp(-|\lambda - \lambda'|T)$ without diffusion and like $\exp(-\lambda'T)$ with diffusion.

The appearance of differences between Lyapunov exponents reflects a relation between forward and backward flow: the stable and unstable manifolds are aligned. A more general difference between forward and backward flow will break this alignment and introduce exponentials in λ'.

Small perturbations in position and in the velocity fields will grow exponentially. Equation (7) gives

$$q_{1,e} = (q_{1,f} - u_1/\lambda')\,e^{-\lambda'T} + u_1/\lambda' \tag{39}$$

$$q_{2,e} = (q_{2,f} + u_2/\lambda')\,e^{\lambda'T} - u_2/\lambda'. \tag{40}$$

The contributions from the displacement to the decay are weaker than in the linear shear flow, since the stretching of the variances is in the same directions as the separation of trajectories, so that an exponential increase in q_2 can be compensated by an exponential increase in variances. However, in the absence of diffusion and for the same Lyapunov exponents in forward and backward directions, an exponentially growing displacement can lead to a drastic drop off, like $\exp(-\exp \lambda t)$, in overlap, simply because the Gaussians are shifted relative to each other.

4. Final remarks

Already, the simple examples in the previous section show a wide range of dynamical behaviour in classical echo experiments. There are two ingredients: the variation in variances, and a displacement between initial and echo densities. Without displacement, the decay in the integrable system is slower than in the chaotic one, and the same applies when there is a displacement in both systems. However, the overlap can drop off faster in an integrable system with displacement compared to a chaotic one without.

The drop off from the displacement is connected with the Gaussian shape of the densities: if the tails fall off more slowly, then the overlap will also decay more slowly as a function of displacement. The shape dependence should be stronger in integrable systems than in chaotic ones, since the stretching of variances and shapes goes in parallel with the exponential growth of displacement.

The calculations are based on Gaussian densities and linearizations of the flow fields near trajectories. This becomes questionable if the densities spread out too far, a problem that occurs more likely and more quickly in chaotic systems than in integrable systems. In integrable systems the dangerous terms come from quadratic dependences of the winding frequencies on action and from curvatures introduced when mapping the tori on to position space. The density will then coil up in whirls. In a 2D freedom system, this can be expected to happen linearly in time. In chaotic systems, the linear approximation is applicable if the stable and unstable manifolds are close to straight lines within the area covered by the density, and if the variations in stretching and contraction rate are small. Typically, this will limit the time interval to a logarithmically short one. For longer times, the density develops tendrils that follow the manifolds as they wiggle through phase space (for an early discussion of such effects within the quantum maps, see Berry *et al* (1979)). In both cases, the degree to which the Gaussian approximation breaks down depends on details of the nonlinear contributions and has to be considered for specific models.

The transformation to a comoving frame eliminates the centre of mass motion and emphasizes the linearized dynamics near the trajectory. The same discussion thus applies

to stationary points. Two examples, shear flow or parabolic type and hyperbolic type, have been discussed here, the third class, elliptic type, has periodically oscillating variances.

2D flows provide an ideal testing ground for the results presented here. Localized spots of dye can be prepared as initial densities and their motion can be followed in various 2D flows (Aref 1984, 2002, Ottino 1990, Homsey 2000, Chaiken *et al* 1986). Elliptic and hyperbolic points can be realized most easily in cellular flows (Jütner *et al* 1997, Williams *et al* 1997, Rothstein *et al* 1999). Molecular diffusivities are fixed by the selection of dye and solvent, but variations of shear rate through the amplitude of the velocity fields and initial variances through the size of the spot provide enough degrees of freedom to explore the full range in behaviour. In particular, the 2D Lorentz force driven flows (Jütner *et al* 1997, Williams *et al* 1997, Rothstein *et al* 1999) should have enough flexibility to study echoes in flows and provide quantitative tests of the various expressions derived here. It should be possible to see the cubic laws in the variances in integrable systems, the chaos assisted spreading in chaotic systems and the dependence on the order of forward/backward propagation.

It would also be of interest to reconsider the experimental protocol of Chaiken *et al* (1986): they followed a closed line of dye. After reversal, the line was displaced left and right of the original trace and also had slightly different widths. This suggests different histories of the centre of mass of small line elements and passage through regions with different degrees of chaos. Using the complete tracings of the flow field as in Voth *et al* (2002), it should be possible to pin down the different regions and interactions and characterize the reversal completely.

Acknowledgment

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